Control statistics for sums of weighted Bernoullis

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1 Problem Statement

Let x_j be N *Bernoulli random variables* each associated with corresponding distribution probability p_j. In other words, x_j takes on the value unity with probability p_j and is zero otherwise. There are also N *weights*, w_j, each $1 \leftrightarrow 1$ with corresponding x_j.

The interest is in the quantity S:

$$S = \sum_{j=0}^{N-1} w_j x_j. \tag{1}$$

and, in particular, its distribution or features of its distribution. These are needed to support choosing a value T such that

$$[\![S > T]\!] \geq 0.90. \tag{2}$$

where $[\![\ldots e \ldots]\!]$ is modern notation for *the probability of the event* e. The criterion of (2) is that of a probabilistic lower bound, specifically, the 0.1 quantile of the cumulative distribution function of S seen as a random variable.

Note that the *Markov inequality*

$$[\![S \geq T]\!] \leq \frac{E[\![S]\!]}{T}. \tag{3}$$

has roughly the same form, but is the wrong bound. What's needed is something like

$$[\![S \geq f(E[\![S]\!], 0.10)]\!] > 0.1. \tag{4}$$

where $f(.,.)$ is some unknown function. Proofs of this inequality such as this for discrete random variables as we have here, adapted from [5].

Assuming $x \geq a$,

$$\begin{aligned} E[\![X]\!] &= (\sum_{x \leq a} x[\![X = x]\!]) + (\sum_{x > a} [\![X = x]\!] \\ &\geq \sum_{x \geq a} a[\![X = x]\!] + 0 \\ &= a \sum_{x \geq a} [\![X = x]\!] \\ &= a[\![X \geq a]\!]. \end{aligned} \tag{5}$$

This doesn't suggest a modification to obtain what's wanted.

The plan, then, is to see if a means can be derived to estimate the 0.1 quantile or, potentially, any other, from moments or cumulants of the distribution of S where these are derived in various ways [4, 19, 7, 14, 17, 12]. The results will be checked against numerical simulations.

This procedure is apparently well known in risk analysis for finance, specifically for calculating *value at risk* [2, 8, 17, 3, 12, 13]. Figure 1 illustrates a *Value at Risk* or *VaR* superimposed on a probability density.

It is attractive to have a closed-form estimate of the quantile for analytical and other reasons.

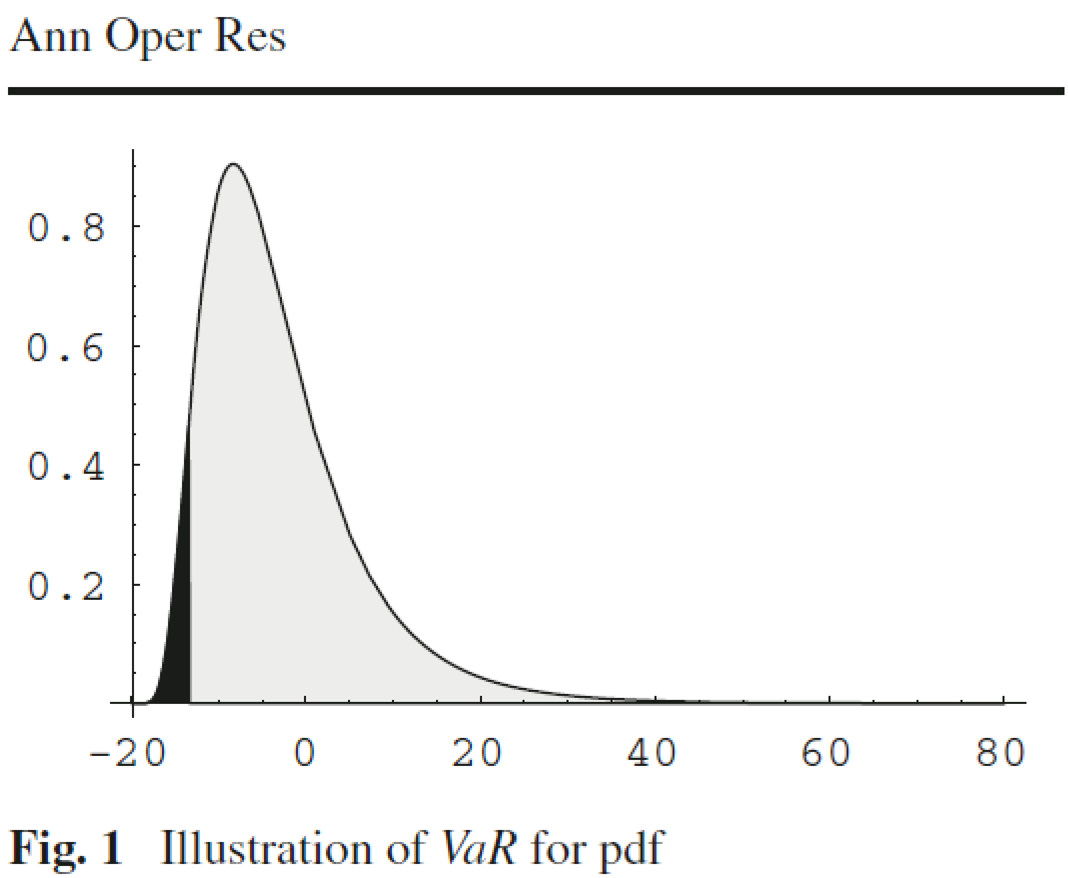

Figure 1: *Value at Risk* or *VaR* superimposed on a p.d.f., as shown in Figure 1 of [3].

2 Using Characteristic Functions, Moments, and Cumulants

The key insight with respect to the sum of independent Bernoulli random variables as (1) presents is[1], from [16, (4.3) of Chapter 9]:

> *...[I]f $X_1, X_2, \ldots, X_n$ are independent random variables whose rth cumulants exist, then the rth cumulant of [their] sum exists and is equal to the sum of the rth cumulations of the individual random variables.* In symbols,
>
> $$\kappa_r[X_1 + \cdots + X_n] = \kappa_r[X_1] + \cdots + \kappa_r[X_n].$$

Of course the same property extends to moments and therefore variances.

The next step, discussed in §3, that of obtaining an estimate of the 0.1 quantile, is to use the Cornish-Fisher expansion [4, 7, 12, 21].

Although cumulants can be obtained from logs of characteristic functions of distributions directly, in this case it is easier to obtain moments first, and then identify cumulants by their direct comparisons [16, Chapter 9, Section 2, and Exercise 2.1].

The core element of (1) is $w_j x_j$, where w_j is a (constant) weight and x_j is a Bernoulli random variable with success probability $[\![x_j = 1]\!] = p_j$. The corresponding characteristic function is

$$\phi_{w_j x_j}(t) = 1 - p_j + e^{\mathrm{i} w_j t} p_j. \tag{6}$$

Here i denotes the imaginary basis for a complex number, or $\sqrt{-1}$.

It is not needed here[2], but note the characteristic function of a some of such elemental characteristic functions is the *product* of the elements [16, Chapter 9, Section 4]:

$$\prod_{j=0}^{N-1} (1 - p_j + e^{\mathrm{i} w_j t} p_j). \tag{7}$$

[1] Emphasis is as in original text. To be consistent with notation here later, "κ" has been substituted for Parzen's "K".

[2] There is a technique for numerically inverting the characteristic function to obtain the corresponding distribution [20]. However that's very heavy-handed, given that all that's wanted is the 0.1 quantile, and it is not at all clear if, in that case, it wouldn't be simpler to simulate and obtain the 0.1 quantile empirically.

Moments of the random variable $X = w_j x_j$ are obtainable from:

$$E[\![X^k]\!] = \frac{1}{\mathrm{i}^k} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \phi_X(0) \tag{8}$$

assuming they exist. Specifically, in this case,

$$\begin{aligned} E[\![(w_j x_j)]\!] &= p_j w_j \\ E[\![(w_j x_j)^2]\!] &= p_j w_j^2 \\ E[\![(w_j x_j)^3]\!] &= p_j w_j^3 \\ E[\![(w_j x_j)^4]\!] &= p_j w_j^4. \\ &\vdots \end{aligned} \tag{9}$$

So, in general,

$$E[\![(w_j x_j)^k]\!] = p_j w_j^k \tag{10}$$

Quoting (2.24) from [16, Chapter 9, Section 2], with slight notational changes, repairing a mistake for $E[\![X^4]\!]$, and adding $E[\![X^5]\!]$ from [15]:

$$\begin{aligned} E[\![X]\!] &= \kappa_1 \\ E[\![X^2]\!] &= \kappa_2 + \kappa_1^2 \\ E[\![X^3]\!] &= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3 \\ E[\![X^4]\!] &= \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4 \\ E[\![X^5]\!] &= \kappa_5 + 5\kappa_4\kappa_1 + 10\kappa_3\kappa_2 + 10\kappa_3\kappa_1^2 + 15\kappa_2^2\kappa_1 + 10\kappa_2\kappa_1^3 + \kappa_1^5. \end{aligned} \tag{11}$$

where κ_k is the kth cumulant. Expressions are available in the reverse direction, too, given in Figure 2, taken from [15].

$$
\begin{aligned}
\kappa_1 &= \mu_1 \\
\kappa_2 &= \mu_2 - \mu_1^2 \\
\kappa_3 &= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3 \\
\kappa_4 &= \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4 \\
\kappa_5 &= \mu_5 - 5\mu_4\mu_1 - 10\mu_3\mu_2 + 20\mu_3\mu_1^2 + 30\mu_2^2\mu_1 - 60\mu_2\mu_1^3 + 24\mu_1^5
\end{aligned}
$$

Figure 2: Cumulants expressed in terms of moments, taken from [15]. Here $\mu_k = E[\![X^k]\!]$.

Actually, there are recurrences connecting moments and cumulants [18]. These are

$$E[\![X^m]\!] = \sum_{l=0}^{m-1} \binom{m-1}{l} \kappa_{m-l} E[\![X^l]\!] \tag{12}$$

connecting moments to cumulants, and

$$\kappa_m = E[\![X^m]\!] - \sum_{l=1}^{m-1} \binom{m-1}{l} \kappa_{m-l} E[\![X^l]\!] \tag{13}$$

connecting cumulants to moments. Both (12) and (13) assume $E[\![X^0]\!] = 1$ and $\kappa_1 = E[\![X]\!]$.

For this case, and using (9) and [15]'s listing of cumulants in terms of moments,

$$\begin{aligned}
\kappa_{1,j} &= p_j w_j. \\
\kappa_{2,j} &= p_j(1-p_j)w_j^2. \\
\kappa_{3,j} &= p_j(p_j-1)(2p_j-1)w_j^3. \\
\kappa_{4,j} &= p_j(1-p_j)(1+6p_j(p_j-1))w_j^4. \\
\kappa_{5,j} &= p_j(p_j-1)(2p_j-1)(1+12p_j(p_j-1))w_j^5.
\end{aligned} \tag{14}$$

Recall from §2 that these are elemental cumulants and, so, to obtain the corresponding cumulants for N pairs of w_j and p_j

$$\kappa_k = \sum_{j=0}^{N-1} \kappa_{k,j}. \tag{15}$$

Note that

- κ_1 corresponds to the *mean*.
- κ_2 corresponds to the *variance*.
- κ_3 corresponds to *skewness*, hereinafter denoted $\mathcal{S}$.
- κ_4 corresponds to *kurtosis*, hereinafter denoted $\mathcal{K}$.
- κ_5 does not correspond to any central moment.

3 Approximating Quantiles

The details of using the Cornish-Fisher expansion with cumulants is concisely summarized on the pertinent *Wikipedia* page [21]. Their presentation is repeated below for easy reference, and is paraphrased.

Given random variable S having first five cumulants κ_1 (mean), κ_2 (variance), κ_3 (skewness), κ_4 (kurtosis), and κ_5, and assuming they exist, S's value, y_q at quantile q can be estimated as

$$y_q \approx \kappa_1 + \eta_q \sqrt{\kappa_2} \tag{16}$$

where $\mathbf{\Phi}^{-1}(.)$ denotes the quantile function of the Gaussian, He_ℓ is the ℓth *probabilists' Hermite polynomial*, and [1, 21]:

$$\begin{aligned} Q &= \mathbf{\Phi}^{-1}(q) \\ \eta_q &= Q + \gamma_1 h_1(Q) + \gamma_2 h_2(Q) + \gamma_1^2 h_{11}(Q) + \gamma_3 h_3(Q) \\ &\quad + \gamma_1 \gamma_2 h_{12}(Q) + \gamma_1^3 h_{111}(Q) + \ldots \end{aligned}$$

$$\gamma_{m-2} = \frac{\kappa_m}{\sqrt{\kappa_2^m}}, m \in \{3,4,5\}$$

$$\begin{aligned} h_1(x) &= \frac{\mathrm{He}_2(x)}{6} \\ h_2(x) &= \frac{\mathrm{He}_3(x)}{24} \\ h_{11}(x) &= -\frac{2\mathrm{He}_3 + \mathrm{He}_1(x)}{36} \\ h_3(x) &= \frac{\mathrm{He}_4(x)}{24} \\ h_{12}(x) &= -\frac{\mathrm{He}_4(x) + \mathrm{He}_2(x)}{24} \\ h_{111}(x) &= \frac{12\mathrm{He}_4(x) + 19\mathrm{He}_2(x)}{324} \end{aligned}$$

4 Numerical Verification of Estimates of Mean and Variance

A simulation of (1) was developed wherein a vector, $\mathbf{p}$, of probabilities were drawn from a Beta distribution, a vector of non-negative weights, $\mathbf{w}$, were drawn from a Gamma distribution, and then, governed by $\mathbf{p}$, $M = 10000$ vectors $\mathbf{x}_k$, $k \in \{1, \ldots, M\}$, drawn from the Bernoulli distribution. Then, per (1), $S_k = \mathbf{w} \cdot \mathbf{x}_k$ was calculated for each and saved as $\{S_k\}$. N, the length of these vectors, was chosen $N = 10$, since that is what is typical for the application described in §1 and §6. An estimated probability density was developed from the M-sized collection of saved sums. Thirty of these are shown in Figure 3.

The first two cumulants, κ_1 and κ_2 of (14) from §2, corresponding to the theoretical mean and variance of $\{S_k\}$, were calculated from the associated $\mathbf{w}$ and $\mathbf{p}$ and then compared to corresponding empirical estimates obtained from $\{S_k\}$. These are shown in Table 1.

The root-mean-square ("r.m.s.") of differences between empirical and theoretical means is 0.101, and between variances is 1.72.

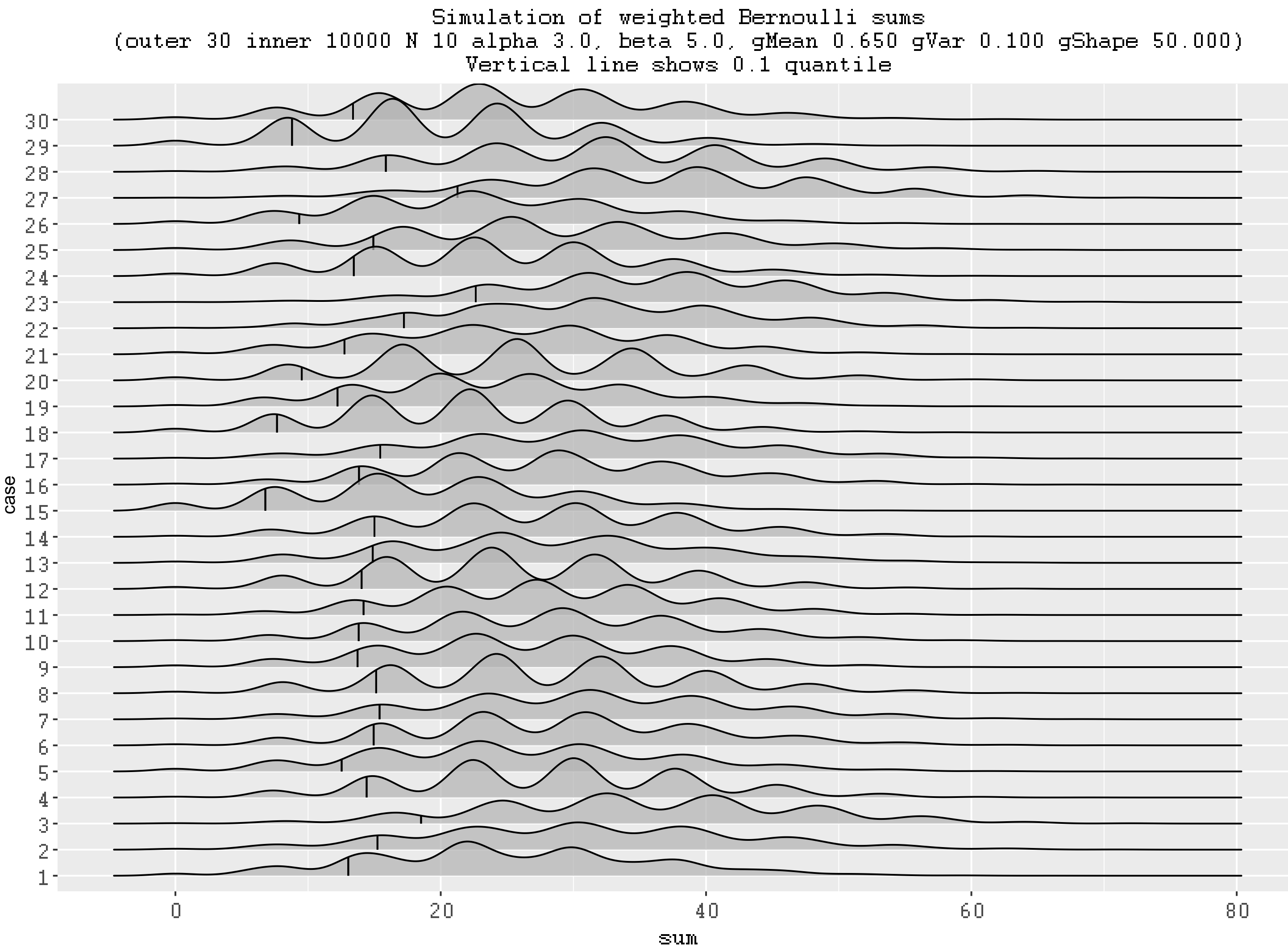

Figure 3: 30 examples of probability densities for randomly chosen probability ranges and weights. Probabilities were drawn from a Beta distribution with shape parameters 3 and 5. Weights were drawn from a Gamma distribution having a shape parameter of 50 and a mean and variance of 0.65 and 0.1, respectively. N was chosen $N = 10$. To obtain the densities, 10000 weighted Bernoulli draws with the chosen weights and probabilities were taken. Note that these cases are *not* the same as those depicted in Table 1.

case	theoretical mean	empirical mean	theoretical variance	empirical variance
1	22.5925	22.7437	115.9253	115.0241
2	25.9221	25.9485	107.8515	107.7594
3	32.6782	32.8422	152.3976	152.4199
4	30.5859	30.6560	121.4982	123.3974
5	29.0620	28.9458	113.5502	114.5186
6	28.3021	28.1854	116.8333	118.4524
7	27.5977	27.5099	111.6372	111.3303
8	31.1029	30.9988	148.6481	148.2248
9	19.9227	19.8461	108.7768	108.6283
10	22.4399	22.3739	117.6045	118.2619
11	29.6900	29.6468	129.0127	127.7211
12	39.4418	39.4884	139.1841	140.3870
13	29.1248	29.1017	130.8515	133.3888
14	31.7461	31.8392	122.5111	122.5502
15	30.3805	30.3650	125.5060	123.6547
16	31.6396	31.6644	150.6599	150.3868
17	32.6584	32.7855	117.2436	118.9896
18	28.5442	28.2846	124.4687	126.5307
19	22.4014	22.5012	125.3310	125.4180
20	34.4542	34.3631	151.0494	150.4672
21	33.6706	33.6410	153.0100	150.4160
22	33.6057	33.8122	133.7792	132.5872
23	29.4692	29.4233	150.3464	150.2598
24	23.7564	23.7159	117.7394	118.4578
25	26.1091	26.1142	106.1848	105.3554
26	31.9851	31.9177	116.6009	113.7822
27	28.1937	28.2118	108.9182	108.8900
28	33.8491	33.5593	126.8576	124.0030
29	33.7225	33.6594	139.3465	142.1042
30	25.0182	24.8609	122.7071	124.4079

Table 1: Comparison of theoretical and empirical values for κ_1 and κ_2. Note that these cases are *not* the same as those depicted in Figure 3.

5 Numerical Verification of Quantile Estimates

This documents calculation of 0.1 point quantiles based upon 60 cases simulated using the same mechanism as in §4, but calculating 20000 Bernoulli sums of (1) in each case. In addition, the `qapx_cf` function from the *PDQutils* package of **R** was used as a check on these calculations, using an algorithm by Lee and Lin [10, 11]. Results are shown in Tables 2 and 3.

case	empirical mean	empirical variance	empirical quantile	theoretical quantile this	theoretical quantile PDQ
1	32.9797	129.8619	16.6621	18.2947	18.1974
2	23.3029	103.9386	11.5903	10.1858	10.0093
3	35.6498	115.6665	21.7294	21.9174	21.8355
4	27.4030	138.5882	13.8070	12.4841	12.2888
5	30.7842	140.5132	16.1893	15.5885	15.4807
6	23.6414	123.5996	8.0337	9.4472	9.2391
7	33.3234	138.2507	17.9625	18.2319	18.0683
8	28.5700	131.2127	15.0152	13.7652	13.6113
9	35.8538	160.2247	17.6493	19.6511	19.5152
10	25.2388	124.8288	10.0660	11.1013	10.9492
11	27.5132	126.8804	14.3114	13.1795	13.0255
12	32.4450	139.9611	15.8156	17.3127	17.1681
13	42.3776	157.4592	25.2537	26.4286	26.3077
14	30.5225	109.8600	14.9775	17.1446	17.0496
15	27.0552	121.9170	14.3961	13.0194	12.8829
16	25.7942	107.2610	13.2263	12.5012	12.3773
17	30.1212	146.6143	14.9276	14.5826	14.4343
18	25.6266	136.2889	8.7289	10.5425	10.3595
19	27.3580	133.6954	14.7043	12.6506	12.5077
20	24.7955	118.6032	11.1641	10.8832	10.7325
21	37.6403	129.6239	22.7020	23.1427	23.0592
22	35.7208	147.4724	20.9749	20.1679	20.0585
23	30.4075	148.2704	15.4425	14.5947	14.4274
24	31.8899	128.0726	16.4501	17.3606	17.2325
25	28.7807	128.3610	14.7281	14.3691	14.2278
26	28.6212	131.7968	14.8376	13.8156	13.6813
27	30.5921	121.9015	14.9505	16.3310	16.2048
28	25.6497	120.8399	12.5376	11.5853	11.3879
29	40.1411	159.3767	23.7778	24.0693	23.9396
30	31.2252	134.8015	15.5482	16.4695	16.3294

Table 2: Comparison of theoretical and empirical values for quantiles, part 1 of 2

The r.m.s. of differences between the empirical 0.1 quantile and the theoretical quantile calculated using the techniques shown here is 1.17. The r.m.s. between the empirical 0.1 quantile and the theoretical calculated using the Lee and Lin algorithm is 1.18 [10, 11]. The empirical quantile was calculated using the `hdquantile` function from the *Hmisc* package of **R**.

case	empirical mean	empirical variance	empirical quantile	theoretical quantile this	theoretical quantile PDQ
31	24.4026	131.0512	8.8306	9.8575	9.6447
32	28.5894	122.9555	14.5983	14.1973	14.0545
33	30.1835	129.4478	15.1990	15.6538	15.5263
34	28.9134	127.6445	15.1286	14.3956	14.2732
35	24.5316	111.6464	13.2815	11.0837	10.9445
36	33.4178	134.7716	19.0582	18.4435	18.3149
37	24.6837	109.9888	11.9591	11.4190	11.3239
38	27.4881	118.6400	14.4048	13.5478	13.4175
39	36.6025	135.8329	22.5416	21.8021	21.6503
40	38.8762	116.2878	23.4947	25.1733	25.0915
41	26.1658	108.4353	14.2998	12.9731	12.8847
42	26.3829	122.5958	13.9600	12.1930	12.0542
43	23.8599	130.3633	8.5535	9.2428	9.0295
44	25.5692	106.7007	14.4374	12.5136	12.4349
45	41.9755	134.9717	27.9455	27.0368	26.9320
46	25.7016	97.7220	13.2841	12.9852	12.8847
47	35.9778	116.9141	22.2802	22.1047	22.0383
48	26.1507	130.0790	13.8580	11.6842	11.5414
49	22.7353	131.6986	8.0267	8.0594	7.8494
50	33.4486	149.3450	17.0829	17.8496	17.6765
51	34.0479	122.0116	20.3035	19.8182	19.6657
52	34.9673	113.4448	20.7247	21.4211	21.3133
53	26.9476	121.0059	14.1009	12.9536	12.8147
54	31.8159	164.3415	16.1803	15.4935	15.3368
55	32.2368	140.2028	16.2697	17.0987	16.9739
56	30.1118	143.8761	15.4801	14.8178	14.6515
57	30.5810	125.7977	15.3738	16.2855	16.1220
58	29.1580	137.4266	14.9654	14.0415	13.8733
59	25.5754	127.4023	10.3064	11.1173	10.9527
60	28.9041	119.2765	14.6813	14.8459	14.7034

Table 3: Comparison of theoretical and empirical values for quantiles, part 2 of 2

6 Application to Original Question

Simulation of the Bernoulli sums from (1) for a large number of cases gives an assortment of empirical probability density functions like those illustrated in Figure 3. The code for calculating theoretical quantiles for these is concise, and is given using **R** as Listing [1] for both the calculation done here and the invocation of the from the *PDQutils* package. Obviously, the latter is more concise, and is therefore recommended if **R** is an option. Cumulants, of course, still need to be calculated.

Otherwise, the calculation done here can be rendered in Python or whatever programming language is suitable. The present implementation assumes the target language supports basic numerical capability such as calculating Hermite polynomials.

Listing [1]

```
library(polynom)
library(orthopolynom)
library(mpoly)
library(moments)
library(PDQutils)

theoreticalMean<- function(W,P) sum(W*P)
theoreticalVariance<- function(W,P) sum(W^2*P*(1-P))
theoreticalSkew<- function(W,P) sum(P*(P-1)*(2*P-1)*W^3)
# (This is not excess kurtosis. Still need to subtract 3 to get kurtosis w.r.t. Gaussian.)
theoreticalKurtosis<- function(W,P) sum(P*(1-P)*(1 + 6*P*(P - 1))*W^4)
theoreticalKappa5<- function(W,P) sum(P*(P-1)*(2*P-1)*(1 + 12*P*(1-P))*W^5)

gamma1<- function(W,P) theoreticalSkew(W,P)/sqrt(theoreticalVariance(W,P)^3)
gamma2<- function(W,P) theoreticalKurtosis(W,P)/sqrt(theoreticalVariance(W,P)^4)
gamma3<- function(W,P) theoreticalKappa5(W,P)/sqrt(theoreticalVariance(W,P)^5)

He.polynomials<- hermite(degree=1:4, kind="he", normalized=TRUE)
ThePoint<- 0.10
TheQuantile<- qnorm(ThePoint)

yAtThePoint<- function(xPoint=0.1, W, P)
{
  #
  xQuantile<- qnorm(xPoint)
  #
  He<- unlist(sapply(X=He.polynomials, FUN=function(He.k) as.function(He.k, silent=TRUE)(xQuantile)))
  #
  h1<- He[2]/6
  h2<- He[3]/24
  h11<- - (2*He[3]+He[1])/36
  h3<- He[4]/120
  h12<- -(He[4] + He[2])/24
  h111<- (12*He[4] + 19*He[2])/324
  #
  g1<- gamma1(W,P)
  g2<- gamma2(W,P)
  g3<- gamma3(W,P)
  #
  mu<- theoreticalMean(W,P)
  sigma<- sqrt(theoreticalVariance(W,P))
  #
  w<- xQuantile + (g1*h1) + (g2*h2 + g1^2*h11) + (g3*h3 + g1*g2*h12 + g1^3*h111)
  #
  yp<- mu + sigma*w
  #
  return(yp)
}

yViaPDQ<- function(Pq, W, P)
{
  cumulants<- c(theoreticalMean(W,P), theoreticalVariance(W,P), theoreticalSkew(W,P),
               theoreticalKurtosis(W,P), theoreticalKappa5(W,P))
  y<- qapx_cf(p=Pq, raw.cumulants=cumulants, support=c(0,Inf), lower.tail=TRUE, log.p=FALSE)
  return(y)
}
```

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 1964, Dover Publications, "26. Probability Functions", 935-936.

[2] Ö. Aktaş, M. Sjöstrand, "Cornish-Fisher expansion and Value-at-Risk method in application to risk management of large portfolios", Masters Thesis, School of Information Science, Computer and Electrical Engineering, Halmstad University, Technical report, IDE1112 , 7 September 2011.

[3] C.-O. Amédée-Manesme, F. Barthélémy, D. Maillard, "Computation of the corrected CornishâĂŞ-Fisher expansion using the response surface methodology: application to *VaR* and *CVaR*", *Annals of Operations Research*, 2018, 1-31.

[4] E. A. Cornish, R. A. Fisher, "Moments and cumulants in the specification of distributions", *Revue de l'Institut International de Statistique*, January 1938, 5(4), 307-320.

[5] C. Coscia, "Math 20 – Inequalities of Markov and Chebyshev", Math 20: Probability, Summer 2018, Dartmouth University.

[6] R. Durrett, *Probability: Theory and Examples*, 4^{th} edition, 2010, Cambridge University Press.

[7] R. A. Fisher, E. A. Cornish, "The percentile points of distributions having known cumulants", *Technometrics*, 1960, 2(2), 209-225.

[8] H. Izhar, "Applying the Cornish-Fisher expansion to value-at-risk estimation in Islamic banking", *Journal of Risk*, 2015, 17(6), 51-72.

[9] S. R. Jaschke "The Cornish-Fisher-Expansion in the context of Delta-Gamma-Normal approximations", SFB 373 Discussion Paper, No. 2001, 54, Humboldt University of Berlin, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, Berlin.

[10] Y.-S. Lee, T.-K. Lin, "Algorithm AS 269: High order Cornish-Fisher expansion", *Journal of the Royal Statistical Society*, Series C (Applied Statistics), 1992, 41(1), 233-240.

[11] Y.-S. Lee, T.-K. Lin, "Correction to Algorithm AS 269: High order Cornish-Fisher expansion", *Journal of the Royal Statistical Society*, Series C (Applied Statistics), 1993, 42(1), 268-269.

[12] D. Maillard, "A users guide to the Cornish Fisher expansion", *SSRN*, Conservatoire National des Arts et Métiers (CNAM), and Amundi Asset Management, May 2018.

[13] D. Maillard, "Computing the Cornish Fisher Expansion moments using Hermite polynomials: A note", *SSRN*, Conservatoire National des Arts et Métiers (CNAM), and Amundi Asset Management, May 2018.

[14] P. McCullagh, J. Kolassa, "Cumulants", in *Scholarpedia*, 2009, 4(3), 4699.

[15] P. McCullagh, "Cumulants", University of Chicago, course STAT306, April 2017.

[16] E. Parzen, *Modern Probability Theory and Its Applications*, 1960, John Wiley & Sons, especially Chapter 9, "Sums of independent random variables".

[17] Risk Management Solutions, "Anatomy of Cornish-Fisher", white paper at *riskconcile.com*, 30^{th} September 2016.

[18] P. J. Smith, "A recursive formulation of the old problem of obtaining moments from cumulants and vice versa", *The American Statistician*, 1995, 49(2), 217-218.

[19] D. L. Wallace, "Asymptotic approximations to distributions", *The Annals of Mathematical Statistics*, September 1958 29(3), 635-654.

[20] L. A. Waller, B. W. Turnbull, J. M. Hardin, "Obtaining distribution functions by numerical inversion of characteristic functions with applications", 1995, *The American Statistician*, 49(4), 346-350.

[21] Wikipedia contributors, "Cornish-Fisher expansion", *Wikipedia, The Free Encyclopedia*, (accessed April 17, 2019).